Superposition of Dependent Sources is Valid in Circuit Analysis

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Abstract— **Circuits texts state or imply that superposition of dependent or controlled sources cannot be used in circuit analysis. I contend that this is a misconception. I present a proof to support my contention and give three examples which illustrate the correct method.**

I. Introduction

In teaching electronics classes, I write most of my circuit equations using superposition. When I apply superposition to circuits containing dependent or controlled sources, students always raise their hands and tell me that I cannot do it. When I ask why, they tell me that is what they were taught in their circuits classes. This has motivated me to investigate how superposition is presented in texts. I surveyed nine introductory books on circuit analysis. One was copyrighted in the 1960s, two in the 1980s, and five in the early 1990s. Six explicitly state that dependent sources must remain active when applying superposition. Three specifically refer to the sources as independent in stating the principle of superposition. Two present an example circuit containing a dependent source which is never set to zero in the superposition. From this limited survey, it is clear that circuits texts either state or imply that superposition of dependent sources is not allowed. I contend that this is a misconception.

To apply superposition to dependent sources, it must be done correctly. The controlling variables of a source must not be set to zero when the source is zeroed. This is illustrated in the following with several examples. In all cases, it is assumed that the circuit is linear and that it contains no nodes at which the voltage is indeterminate and no branches in which the current is indeterminate. This specifically rules out cases where current sources are connected in series or voltage sources are connected in parallel.

II. The Principle of Superposition

A general proof of the principle of superposition is presented by Scott [1], where he takes all sources to be independent. With some modifications, the proof presented here follows Scott's. The starting point is to assume a general set of mesh or node equations for any given linear circuit. Node equations are assumed here. If the circuit contains voltage sources, they must first be converted into equivalent current sources by making Norton equivalent circuits. Such a transformation does not change the response of the network external to the source. If the current response in the source branch is to be calculated, a mesh equation analysis must be used. In the case of a voltage source with no series resistance, a transformation which Scott calls "pushing a voltage source through a node" must first be performed. This replaces a voltage source having no series resistance with several voltage sources, one in each of the branches radiating from the node to which the original source connects.

The general node equations for any linear circuit containing n nodes can be written

$$\begin{aligned} i_1 &= +y_{11}v_1 - y_{12}v_2 \cdots - y_{1n}v_n \\ i_2 &= -y_{12}v_1 + y_{22}v_2 \cdots - y_{2n}v_n \\ &\vdots \\ i_n &= -y_{1n}v_1 - y_{2n}v_2 \cdots + y_{nn}v_n \end{aligned} \tag{1}$$

where i_j is the net current delivered to node j by independent and dependent sources, v_j is the voltage at node j, y_{jj} is the total admittance radiating from node j, and y_{ij} is the admittance between nodes i and j. The node equations can be written in the matrix form $i = yv$, where y is an admittance matrix. Because the dependent sources are contained in the current matrix i, the y matrix corresponds to what is called the branch admittance matrix of the circuit. This is the admittance matrix with all dependent sources set to zero. This matrix is symmetrical.

A determinant solution for v_1 can be written

$$v_1 = \frac{1}{\Delta}\begin{vmatrix} i_1 & -y_{12} & \cdots & -y_{1n} \\ i_2 & y_{22} & & -y_{2n} \\ \vdots & & & \\ i_n & -y_{2n} & \cdots & y_{nn} \end{vmatrix} \tag{2}$$

where Δ is the determinant

$$\Delta = \begin{vmatrix} y_{11} & -y_{12} & \cdots & -y_{1n} \\ -y_{12} & y_{22} & & \\ \vdots & & & \\ -y_{1n} & -y_{2n} & \cdots & y_{nn} \end{vmatrix} \tag{3}$$

When a cofactor expansion of (2) is made, the solution for v_1 can be written

$$v_1 = i_1\frac{\Delta_{11}}{\Delta} - i_2\frac{\Delta_{12}}{\Delta} + i_3\frac{\Delta_{13}}{\Delta} - \cdots \tag{4}$$

where Δ_{ij} is the determinant formed by deleting row i and column j in Δ.

Each term in (4) is identical to the term which would be written if only the source which generates that term is active and all other sources are zero. Thus the total response is written as the sum of the responses obtained with each source acting alone. This proves the principle of superposition. Because no assumption is made on the type of any source, it follows that the principle can be applied to both independent and dependent sources. When dependent sources are present, however, the solution contains unknowns in the terms on the right side of the equation. Thus the solution of simultaneous equations may be required. This is illustrated in the examples below.

It is important to note that the above proof does not imply that the controlling variables of a dependent source are set to zero when applying superposition to the source. Only the output of the source is set to zero. When this is done correctly, it is possible to write the equations for the response of a circuit by considering only one source at a time.

III. Examples

Solutions to the following three example circuits containing dependent sources are all written by inspection using superposition. I invite the reader to work the examples the way circuits books teach by applying superposition without zeroing the dependent sources. It is probably easier not to use superposition at all.

A. *Example 1*

The object in this example is to solve for the current i_1 in the circuit shown in Figure 1. By superposition, we can write

$$\begin{aligned} i_1 &= \frac{30}{6+4+2} + 3\times\frac{4}{6+4+2} - 8i_1\times\frac{6}{6+4+2} \\ &= 3.5 - 4i_1 \end{aligned}$$

The current i_1 occurs on both sides of the equation. Solution yields

$$i_1 = \frac{3.5}{5} = 0.7 \text{ A}$$

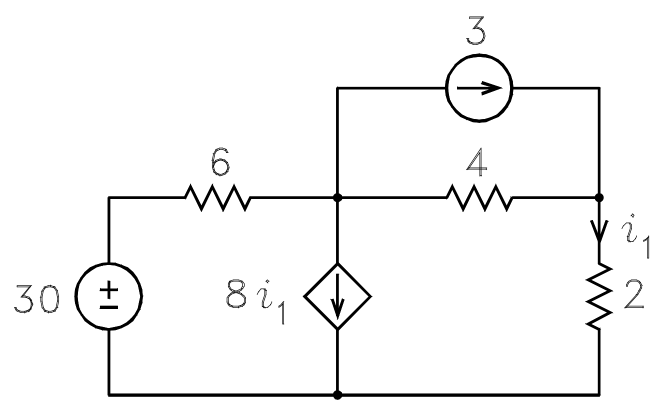

Fig. 1. Example circuit 1.

B. *Example 2*

The object in this example is to solve for the Thevenin equivalent circuit seen looking into the terminals $A - A'$ of the circuit shown in Figure 2(a). First we solve for the open-circuit output voltage. Because this is a function of v_x, we must first solve for v_x. By superposition, we can write

$$v_x = 3 \times 2\|40 + 5v_x \times \frac{2}{40+2}$$

Solution yields

$$v_x = \frac{3 \times 2\|40}{1 - 10/42} = 7.5 \text{ V}$$

By superposition, the open-circuit output voltage is given by

$$v_{oc} = 3 \times 2\|40 - 5v_x \times \frac{40}{40+2} = -30 \text{ V}$$

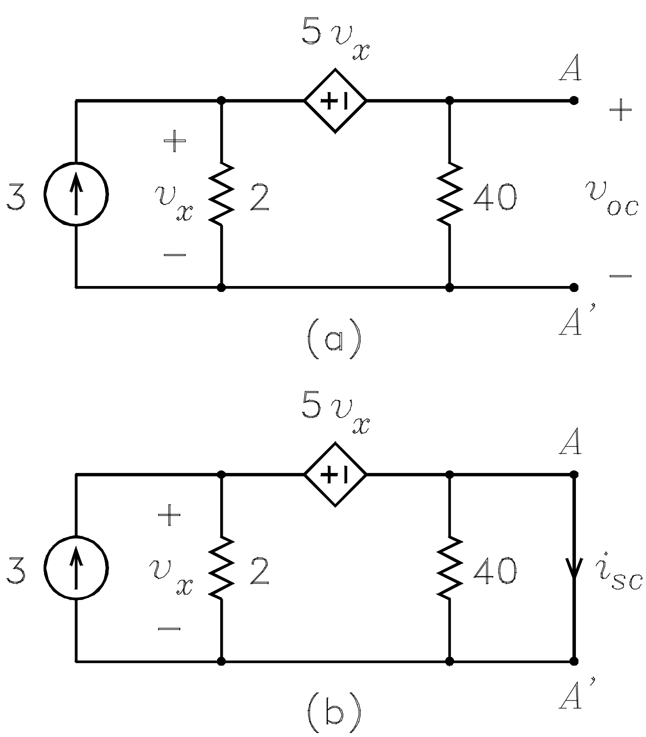

Fig. 2. Example circuit 2.

To solve for the output resistance, we use the equation $r_{out} = v_{oc}/i_{sc}$, where i_{sc} is the short circuit output current. The circuit for i_{sc} is given in Figure 2(b). By superposition, we have

$$v_x = 3 \times 0 + 5v_x$$

Solution yields $v_x = 0$. It follows that

$$i_{sc} = 3 \text{ A}$$

Thus the output resistance is

$$r_{out} = \frac{v_{oc}}{i_{sc}} = \frac{-30}{3} = -10 \text{ -}$$

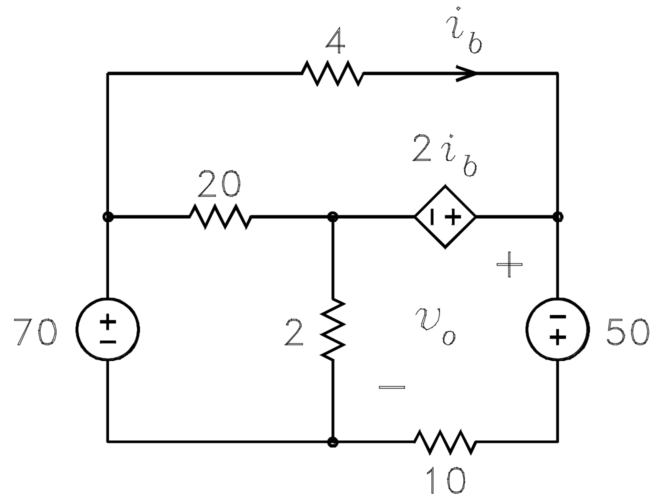

Fig. 3. Example circuit 3.

C. Example 3

The object in this example is to solve for the voltage v_o in the circuit shown in Figure 3. First, we solve for i_b. By superposition, we can write

$$\begin{aligned} i_b &= \frac{70}{4\|20 + 2\|10} \times \frac{20}{4+20} \\ &\quad + \frac{50}{10 + 4\|20\|2} \times \frac{20\|2}{4 + 20\|2} \\ &\quad - \frac{2i_b}{20\|2 + 4\|10} \times \frac{10}{4+10} \\ &= \frac{35}{3} + \frac{25}{18} - \frac{11}{36} i_b \end{aligned}$$

Solution yields

$$i_b = \frac{35/3 + 25/18}{1 + 11/36} = 10 \text{ A}$$

By superposition, v_o is given by

$$\begin{aligned} v_o &= 70 \times \frac{2\|10}{4\|20 + 2\|10} - 50 \times \frac{4\|20\|2}{10 + 4\|20\|2} \\ &\quad + 2i_b \times \frac{4\|10}{20\|2 + 4\|10} \\ &= \frac{70}{3} - \frac{50}{9} + \frac{11}{9} \\ &= 30 \text{ V} \end{aligned}$$

IV. Conclusion

Superposition of dependent sources is a valid method of writing equations for circuits. If anyone can show me a circuit where it cannot be done, I maintain that the circuit contains a node at which the voltage is indeterminate or a branch in which the current is indeterminate. In this case, the circuit cannot be analyzed by any other means.

References

[1] R. E. Scott, *Linear Circuits*, New York: Addison-Wesley, 1960.